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TWO DIMENSIONAL QCD IS A ONE DIMENSIONAL KAZAKOV-MIGDAL MODEL

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Abstract

We calculate the partition functions of QCD in two dimensions on a cylinder and on a torus in the gauge $\partial_0 A_0 = 0$ by integrating explicitly over the non zero modes of the Fourier expansion in the periodic time variable. The result is a one dimensional Kazakov-Migdal matrix model with eigenvalues on a circle rather than on a line. We prove that our result coincides with the standard expansion in representations of the gauge group. This involves a non trivial modular transformation from an expansion in exponentials of g^2 to one in exponentials of $1/g^2$. Finally we argue that the states of the $U(N)$ or $SU(N)$ partition function can be interpreted as a gas of N free fermions, and the grand canonical partition function of such ensemble is given explicitly.

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1 Introduction

The last year has seen a revival of attempts to understand non-perturbative QCD using the $1/N$ expansion . First, Kazakov and Migdal (KM) [1] proposed a model with an adjoint multiplet of scalar fields coupled to external gauge fields, which turned out [2] to be exactly solvable in the large N limit. The model was originally proposed as a way of finding the master field that would solve four dimensional QCD in the large N limit. It is not clear yet whether this goal will be achieved or not, as it is almost certain that some modifications to the original model are needed for it to admit the QCD fixed point in the continuum limit, but the proposal has already generated a considerable literature (see for example [3] and references therein).

It has been pointed out [4] that the d-dimensional KM model with a quadratic potential describes the high temperature limit of pure QCD in $d+1$ dimensions; however, for $d > 1$ such theory does not have a continuum limit unless higher order terms are added to the potential. On one hand, this is in agreement with known results in dimensional reduction of finite temperature QCD, but, on the other hand, higher terms in the potential are difficult to control and their arbitrariness limits the predictive power of the model.

Whatever its relationship with four-dimensional QCD, it is remarkable that the KM model is the only known example of a matrix model, hence of a string theory, that is solvable in more than one dimension in the large N limit. The question of what kind of string theory corresponds to the KM model has also been addressed recently [5]. It turns out that the KM model describes, at least in the large N limit, a string theory with infinite string tension, where the string has collapsed into a branched polymer. This might explain the solvability of the KM model even in dimensions higher than one.

More recently, in quite a different context, Gross and Taylor [6] proved that QCD in two dimensions (QCD2) can be interpreted as a string theory in the large N limit, by showing that the coefficients of the expansion of the partition function of QCD2 in power series of $1/N$ can be interpreted in terms of mappings from a two dimensional surface onto a two dimensional target space. A lot of information on the underlying string theory is encoded in the coefficients of this expansion, but the lack of a prescription for the string action and for the calculation of the string path integral limits the efforts to gather non-perturbative information or extrapolate to higher dimensions.

In this paper, we proceed in a different direction in the understanding of the string theory underlying QCD2, by proving that QCD2 on a cylinder and on a torus is described by a matrix model which is exactly a one dimensional KM model on a circle with one extra condition: that each eigenvalue of the scalar field in the KM model be defined modulo integer multiples of 2π. In short, QCD2 on these surfaces is a periodic one-dimensional KM model, and dimensional reduction (or equivalently the high temperature expansion) is exact in this case.

We emphasize that we are not taking the large N limit, and our proof is valid for any N. Furthermore, the method we employ leads us to find expressions for the partition functions that involve exponentials of the *inverse* gauge coupling, unlike the usual character expansions [7, 8, 9]. The existence of such expressions is a consequence of the well-known fact that the partition functions of non-abelian gauge theories on two-dimensional surfaces with boundaries are kernels of the heat equation on the gauge group manifold, as expected from the path integral formulation. Such kernels, at least for $U(N)$ and $SU(N)$, are expected to admit representations in terms of periodic gaussians in the invariant angles (sometimes called Weyl angular parameters, they are just i times the logarithm of the eigenvalues of the unitary matrix in the fundamental representation). Indeed, such a representation has been known for a while in the case of the partition function on the disk [10, 11].

By a careful gauge-fixing procedure, we are able to find a similar representation for the kernel associated with the cylinder, and this leads to the partition function on the torus via identification of the boundaries and group integration. The invariant angles that parameterize our representation are precisely the eigenvalues of the KM model, and are directly related to the Polyakov loops around the cylinder, as expected from [4].

After deriving the partition functions, we are able to explicitly prove the equivalence of our expressions with the corresponding character expansions, by performing a modular transformation that generalizes to non-abelian gauge theories the modular inversion of Jacobi's θ_3, which in fact is the partition function of QED on a torus.

As a result of our analysis we also find a generating function for the partition functions of $U(N)$ $[SU(N)]$. As expected from the identification of QCD2 with a one dimensional matrix model, this grand canonical partition function is given in terms of infinite products of free fermion partition functions.

The paper is organized as follows: in section 2, after a few general remarks, we calculate the partition function on the cylinder with an appropriate choice of gauge fixing, and we get a representation in terms of angles that coincides with the announced KM model. In section 3 we proceed to reconstruct the character expansion on the cylinder by a modular transformation. In section 4 we calculate the partition function on the torus and we find the generating function that reproduces the partition functions of $SU(N)$ $(U(N))$ for any N. In section 5 we present our conclusions, and finally in the appendix we present an alternative derivation of the kernel on the cylinder, following closely the techniques applied to hermitian matrix models in [12].

2 The Partition Function on the Cylinder

It is well known by now that two dimensional QCD defined on a manifold $\mathcal{M}$ of genus G and with a metric $g_{\mu\nu}$ is exactly solvable. The partition function is given by

$$\begin{aligned}
\mathcal{Z}_{\mathcal{M}}(N,\mathcal{A}) &= \int \mathcal{D}A_\mu e^{-\frac{1}{4\tilde{g}^2}\int_{\mathcal{M}} d^2x\sqrt{g}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}} \\
&= \sum_R d_R^{2-2G} e^{-\frac{1}{2}\mathcal{A}\tilde{g}^2 C_2(R)},
\end{aligned} \tag{1}$$

where the sum is over all equivalence classes of irreducible representations R, d_R is their dimension and $C_2(R)$ is the quadratic Casimir in the representation R. Similarly, the heat kernel defined by a surface of genus G and n boundaries is given by [8, 9]

$$\mathcal{K}_{G,n}(g_1,\ldots,g_n;N,\mathcal{A}) = \sum_R d_R^{2-2G-n}\chi_R(g_1)\cdots\chi_R(g_n)e^{-\frac{1}{2}\mathcal{A}\tilde{g}^2C_2(R)}, \tag{2}$$

where g_i are the Wilson loops evaluated along the boundaries, and χ_R denotes the Weyl character of the representation R. For dimensional reasons, and because of the invariance of the action under area preserving diffeomorphisms, eqs. (1) and (2) depend only on the variable $\tilde{g}^2\mathcal{A}$. We will henceforth denote this variable by t.

The heat kernels in eq. (2) are class functions, and therefore they must admit a representation in terms of the eigenvalues of the group elements g_i in the fundamental representation, which are pure phases for $SU(N)$ and $U(N)$. An explicit expression in terms of these phases was found by Menotti and Onofri [10] for the simplest case, the disk. It is of the form

$$\begin{aligned}
\mathcal{K}_{0,1}(\phi_i,t) &= \mathcal{N}(t)\sum_{\{l_i\}=-\infty}^{+\infty}\prod_{i<j=1}^{N}\frac{\phi_i-\phi_j+2\pi(l_i-l_j)}{2\sin\frac{1}{2}\left[\phi_i-\phi_j+2\pi(l_i-l_j)\right]}\times \\
&\times \exp\left[-\frac{1}{t}\sum_{i=1}^{N}\left(\phi_i+2\pi l_i\right)^2\right].
\end{aligned} \tag{3}$$

In order to formulate QCD2 as a KM model, we will construct a similar representation for the kernel defined by the cylinder ($\mathcal{K}_{0,2}$), which we will denote simply by $\mathcal{K}_2$. It will be convenient to work in first order formalism, where the action is given by

$$S(N,t) = \frac{2}{t}\int_0^{2\pi} dx d\tau\ \mathrm{Tr}[F^2 - iFf(A)], \tag{4}$$

where

$$f(A) = \partial_0 A_1 - \partial_1 A_0 - i[A_0,A_1], \tag{5}$$

while F and A are considered as independent fields. By using the invariance under area preserving diffeomorphisms of the original action, we have restricted the metric in eq. (5) to be flat and rescaled space-time coordinates (x,τ) so that they both range in the interval $(0,2\pi)$. The variable t appears then explicitly at the r.h.s.

In the case of a cylinder we have to fix the value of the Polyakov loop at the two boundaries (say at $x = 0$ and $x = 2\pi$) by introducing suitable delta functions :

$$\begin{aligned} \mathcal{K}_2(g_1, g_2, t) &= \int \mathcal{D}A_\mu \mathcal{D}F \exp\left(-\frac{2}{t}\mathrm{Tr}\int_0^{2\pi} dx d\tau [F^2 - iFf(A)]\right) \times \\ &\times \ \hat{\delta}\left(W(0), g_1\right) \hat{\delta}\left(W(2\pi), g_2\right) \psi(g_1)\psi(g_2), \end{aligned} \tag{6}$$

where

$$W(x) = \mathcal{P}e^{i\int_0^{2\pi} d\tau A_0(x,\tau)}, \tag{7}$$

and $\hat{\delta}(g,h)$ denotes the conjugation invariant delta function on the group manifold, defined by

$$\hat{\delta}(g,h) = \int dU \delta(UgU^\dagger h). \tag{8}$$

The factors $\psi(g_1)$ and $\psi(g_2)$ are just normalization factors; they depend only on the eigenvalues of g_1 and g_2 and they will be chosen so that the sewing of two cylinders corresponds to just a group integration:

$$\int \mathcal{K}_2(g_1, g,)\mathcal{K}_2(g^\dagger, g_2) dg = \mathcal{K}_2(g_1, g_2). \tag{9}$$

All fields in eq. (6) are periodic with period 2π in τ and can be expanded in Fourier series as

$$\left[\begin{array}{c} A_0(x,\tau) \\ A_1(x,\tau) \\ F(x,\tau) \end{array}\right] = \sum_{n=-\infty}^{+\infty} \left[\begin{array}{c} B_n(x) \\ A_n(x) \\ F_n(x) \end{array}\right] e^{in\tau}. \tag{10}$$

All the Fourier modes with $n \neq 0$ can be integrated away in eq. (6) with a suitable gauge choice, namely

$$\partial_0 A_0(x,\tau) = 0 \quad \leftrightarrow \quad B_n(x) = 0, \quad n \neq 0. \tag{11}$$

In this gauge the kernel becomes

$$\begin{aligned} \mathcal{K}_2(g_1, g_2, t) &= \int \mathcal{D}B_0 \prod_n \mathcal{D}A_n \mathcal{D}F_n e^{-\frac{1}{t}\mathrm{Tr}\int_0^{2\pi} dx \mathcal{L}(B_0, A_n, F_n)} \Delta_{FP} \times \\ &\times \ \hat{\delta}\left(W(0), g_1\right) \hat{\delta}\left(W(2\pi), g_2\right) \psi(g_1)\psi(g_2), \end{aligned} \tag{12}$$

where

$$\mathcal{L} = \sum_{n=-\infty}^{+\infty} \{F_n F_{-n} - nF_n A_{-n} + i\delta_{n,0} F_0 \partial_1 B_0 - F_n[B_0, A_{-n}]\}, \tag{13}$$

while Δ_{FP} is the Faddeev-Popov determinant

$$\Delta_{FP} = \prod_{n\neq 0} det \frac{\delta[n\varepsilon_n - [B_0, \varepsilon_n]]_{rs}}{\delta\varepsilon_{n,tl}} \tag{14}$$

and $W(x)$ is now given by

$$W(x) = e^{2i\pi B_0(x)}. \tag{15}$$

At this point the functional integration over A_n $(n \neq 0)$ leads to a product of delta functions :

$$\prod_{n\neq 0} \delta\left(nF_n - [B_0, F_n]\right) = \frac{1}{\Delta_{FP}} \prod_{n\neq 0} \delta(F_n). \tag{16}$$

By inserting this equation in eq. (12) we obtain

$$\begin{aligned} \mathcal{K}_2(g_1, g_2) &= \int \mathcal{D}B\mathcal{D}A\mathcal{D}F e^{-\frac{4\pi}{t}\mathrm{Tr}\int_0^{2\pi} dx\left(F^2 + iF\partial B - F[B,A]\right)} \times \\ &\times \ \hat{\delta}\left(W(0), g_1\right) \hat{\delta}\left(W(2\pi), g_2\right) \psi(g_1)\psi(g_2) \\ &= \int \mathcal{D}B\mathcal{D}A e^{-\frac{\pi}{t}\mathrm{Tr}\int_0^{2\pi} dx[\partial B - i[A,B]]^2} \times \\ &\times \ \hat{\delta}\left(W(0), g_1\right) \hat{\delta}\left(W(2\pi), g_2\right) \psi(g_1)\psi(g_2). \end{aligned} \tag{17}$$

where we have replaced B_0, A_0 and F_0 with B, A and F. This result can be interpreted as follows: the action is a KM model in one continuum dimension with boundary conditions depending on $e^{2i\pi B}$ rather than B; this implies that if, for instance, $B(0)$ satisfies the boundary conditions imposed by the delta functions at $x = 0$, then the latter are satisfied by any other matrix whose set of eigenvalues coincides with the one of $B(0)$ *modulo integer numbers*. It is indeed possible to restrict all the eigenvalues of $A_0(x,\tau) = B(x)$ to be in the interval $[0, 1)$ by a further gauge choice. In fact if $A_0(x,\tau) = U(x)\mathrm{diag}\left(\lambda_i(x)\right) U^\dagger(x)$ then the gauge transformation generated by the unitary matrix $h(x,\tau) = U(x)\mathrm{diag}\left(e^{in_i\tau}\right) U^\dagger(x)$ is periodic in τ with period 2π, preserves the gauge choice (11), and just amounts to a shift of all the eigenvalues $\lambda_i(x)$ by the integer numbers n_i. It is clear though that to avoid discontinuities in the gauge transformation the integers n_i in $h(x,t)$ have to be x independent , so the eigenvalues $\lambda_i(x)$ can be restricted to the interval $[0, 1)$ only for one particular value of x, say $x = 0$. By means of techniques which are by now standard in matrix models, the matrix B can be diagonalized and the action written in terms of its eigenvalues $\lambda_i(x)$, as

$$\begin{aligned} \mathcal{K}_2(g_1, g_2, t) &= \int \mathcal{D}\lambda_i(x)\Delta\left(\lambda(0)\right)\Delta\left(\lambda(2\pi)\right)\exp\left\{-\frac{\pi}{t}\int_0^{2\pi} dx\partial\lambda_i(x)\partial\lambda^i(x)\right\} \times \\ &\times \int dU_1 dU_2 \delta\left[U_1 e^{2i\pi\lambda(0)}U_1^\dagger e^{-i\theta}\right]\delta\left[U_2 e^{2i\pi\lambda(2\pi)}U_2^\dagger e^{-i\phi}\right]\psi(\theta)\psi(\phi), \end{aligned} \tag{18}$$

where we inserted the explicit expression for the δ functions, we denoted by $\lambda(0)$, $\lambda(2\pi)$, θ, ϕ the diagonal matrices whose eigenvalues are respectively $\lambda_i(0)$, $\lambda_i(2\pi)$, θ_i, and ϕ_i, and we chose $e^{-i\theta}$ [resp $e^{-i\phi}$] to denote the diagonal form of g_1 [resp of g_2]. $\Delta(\lambda)$ is the Vandermonde determinant of a hermitian matrix with eigenvalues λ_i.

The integrals over the unitary matrices U_1 and U_2 can be calculated from the identity

$$\begin{aligned} \exp\left[\beta\mathrm{Tr}(V + V^\dagger - 2)\right] &= \sum_R d_R \beta^{-N^2/2}\chi_R(V)\left(1 + O(1/\beta)\right) \\ &= \beta^{-N^2/2}\delta(V), \end{aligned} \tag{19}$$

where V is a unitary matrix.

Inserting this expression of $\delta(V)$ in eq. (8), and using the saddle point method to calculate the integral for large β[1], we find

$$\int dU \delta\left(U e^{i\sigma} U^{\dagger} e^{i\xi}\right) = \sum_{P} \sum_{\{n_j\}=-\infty}^{+\infty} \frac{(-1)^{P+(N-1)\sum_k n_k}}{J(\sigma)J(\xi)} \prod_{j=1}^{N} \delta(\sigma_j + \xi_{P(j)} + 2\pi n_j), \quad (20)$$

where P denotes a permutation of indices,

$$J(\sigma) = \prod_{i<j} 2 \sin \frac{\sigma_i - \sigma_j}{2} \quad (21)$$

is the Vandermonde determinant for a unitary matrix, and we made use of the fact that $J(\sigma + 2\pi n) = (-1)^{(N-1)\sum_j n_j} J(\sigma)$.

The quadratic functional integral over the eigenvalues can be performed by a ζ-function regularization of the divergences, leading to

$$\begin{aligned} &\int \mathcal{D}\lambda_i(x) \exp\left[-\frac{\pi}{t}\int_0^{2\pi} dx \sum_{i=1}^{N} (\partial\lambda_i(x))^2\right] \\ &= \left(\frac{1}{4\pi^2 t}\right)^{N/2} \exp\left[-\frac{1}{2t}\sum_{i=1}^{N} (\lambda_i(0) - \lambda_i(2\pi))^2\right]. \end{aligned} \quad (22)$$

The result at the r.h.s. of eq. (22) is the same that we would have obtained if we had started from a KM model on a one dimensional lattice with an arbitrary number of sites in the interval $[0, 2\pi)$. The result is independent of the number of sites due to the remarkable scaling properties of the KM model, already noticed in [14]. By substituting eq. (22) and eq. (20) into eq. (18) one finally obtains

$$\begin{aligned} \mathcal{K}_2(g_1, g_2, t) \;=\; &\sum_P \frac{t^{-N/2}}{J(\theta)J(\phi)} \sum_{\{l_i\}} (-1)^{P+(N-1)\sum_j l_j} \\ &\exp\left[-\frac{1}{2t}\sum_{i=1}^{N}\left(\phi_i - \theta_{P(i)} + 2\pi l_i\right)^2\right]. \end{aligned} \quad (23)$$

Here the normalization factor has also been determined. It is given by $\psi(g) = J(g)/\Delta(g)$, where $\Delta(g) = \prod_{i<j}(\theta_i - \theta_j)$ if the eigenvalues of g are denoted by $e^{i\theta_i}$.

For the gauge group $U(N)$ the eigenvalues ϕ_i, θ_i and the integers l_i are unconstrained, whereas for $SU(N)$ we can choose $\sum_i \phi_i = \sum_i \theta_i = 0$, and this in turn constrains the integers l_i to obey $\sum_i l_i = 0$. In fact if we choose the gauge $\lambda_i(0) = \theta_i$, as previously discussed, then the fact that $\lambda_i(2\pi) = \phi_i + 2\pi l_i$, together with the continuity of the function $\sum_i \lambda_i(x)$, implies $\sum_i l_i = 0$. Thus for $SU(N)$ the sign factor in each term of eq. (23) is just given by $(-1)^P$. Note also that for $SU(N)$ the factor $t^{-N/2}$ must be replaced by $t^{(1-N)/2}$.

[1] These generalizations of the Harish-Chandra integral are discussed in section 3 of ref. [13]

3 Modular Inversion of the Kernel on the Cylinder

The expression for the kernel on a cylinder obtained in the previous section, as announced, involves exponentials in $1/t$, unlike eq. (2). As a preliminary check on eq. (23), one can easily verify that by taking the limits $\theta \to 0$ or $\phi \to 0$ one obtains the kernel on the disk, eq. (3).

To verify that the two expressions are related by a modular transformation, let us consider now eq. (23) for the gauge group $SU(N)$:

$$\begin{aligned} \mathcal{K}_2(g_1, g_2, t) &= \frac{1}{2\pi N!} \sum_{P,P'} \frac{t^{(1-N)/2}}{J(\theta)J(\phi)} \sum_{\{l_i\}} \int_0^{2\pi} d\beta (-1)^{P+P'} \\ &\exp\left(-\frac{1}{2t}\sum_{i=1}^{N}\left[\left(\phi_{P'(i)} - \theta_{P(i)} + 2\pi l_i\right)^2 - 2i\beta t l_i\right]\right). \end{aligned} \tag{24}$$

The lagrange multiplier β has been introduced to impose the condition $\sum l_i = 0$, and a redundant double sum over permutations has replaced the simple sum of eq. (23).

By completing the square, eq. (24) can be rewritten as

$$\begin{aligned} \mathcal{K}_2(g_1, g_2, t) &= \frac{1}{2\pi N!} \sum_{P,P'} \frac{t^{(1-N)/2}}{J(\theta)J(\phi)} \sum_{\{l_i\}} \int_0^{2\pi} d\beta (-1)^{P+P'} \\ &\exp\left(-\frac{2\pi^2}{t}\sum_{i=1}^{N}\left(l_i + \frac{\phi_{P'(i)} - \theta_{P(i)}}{2\pi} - i\frac{\beta}{4\pi^2}t\right)^2\right) \times \\ \times &\exp\left(-\frac{Nt\beta^2}{8\pi^2}\right), \end{aligned} \tag{25}$$

so that one can use the well known modular transformation of the function θ_3,

$$\sum_{l=-\infty}^{+\infty} \exp\left(-\frac{(\theta + l)^2}{4t}\right) = \sum_{n=-\infty}^{+\infty} \exp\left(-4\pi^2 n^2 t\right) \exp(2\pi i n\theta)(4\pi t)^{1/2}, \tag{26}$$

to obtain

$$\begin{aligned} \mathcal{K}(g_1, g_2, t) &= \frac{1}{2\pi N!} \sum_{P,P'} \frac{t^{1/2}}{J(\theta)J(\phi)} \sum_{\{n_i\}} \int_0^{2\pi} d\beta (-1)^{P+P'} \times \\ &\times \exp\left[-\frac{t}{2}\sum_{i=1}^{N} n_i^2 + 2\pi i \sum_{i=1}^{N} n_i \left(\frac{\phi_{P'(i)} - \theta_{P(i)}}{2\pi} - i\frac{\beta}{4\pi^2}t\right) - \frac{Nt\beta^2}{8\pi^2}\right] \\ &= \frac{t^{1/2}}{2\pi N!} \sum_{\{n_i\}} \exp\left(-\frac{t}{2}\left[\sum_i n_i^2 - \frac{1}{N}\left(\sum_i n_i\right)^2\right]\right) \times \end{aligned}$$

$$
\begin{aligned}
&\times \frac{\det\{e^{in_i\phi_j}\}}{J(\phi)} \frac{\det\{e^{-in_i\theta_j}\}}{J(\theta)} \times \\
&\times \int_0^{2\pi} d\beta \exp\left[-\frac{Nt}{8\pi^2}\left(\beta - \frac{2\pi}{N}\sum_i n_i\right)^2\right].
\end{aligned} \tag{27}
$$

The sum over $\{n_i\}$ can now be restricted to the region $n_1 > n_2 > \cdots > n_N$, at the expense of the factor $1/N!$. One can further notice that both the first exponential and the determinants are invariant if all the n_i's are shifted by the same constant. One is then lead to define $r_i = n_i - n_N - N$, in terms of which the kernel becomes

$$
\begin{aligned}
\mathcal{K}_2(g_1, g_2, t) &= \frac{\sqrt{t}}{2\pi} \sum_{r_1>r_2>\ldots>r_N=-N} \exp\left\{-\frac{t}{2}\left[\sum_i r_i^2 - \frac{1}{N}\left(\sum r_i\right)^2\right]\right\} \times \\
&\times \frac{\det\{e^{ir_i\phi_j}\}}{J(\phi)} \frac{\det\{e^{-ir_i\theta_j}\}}{J(\theta)} \times \\
&\times \sum_{n_N=-\infty}^{+\infty} \int_{\beta_0}^{\beta_1} d\beta \exp\left[-\frac{Nt}{8\pi^2}\left(\beta - \frac{2\pi}{N}\sum_i r_i\right)^2\right],
\end{aligned} \tag{28}
$$

where $\beta_0 = -2\pi n_N - 2\pi N = \beta_1 - 2\pi$. The sum over n_N simply reconstructs the gaussian integral from $-\infty$ to $+\infty$, that can be trivially calculated. It is easy to recognize now that in eq. (28) the exponential is related to the quadratic Casimir of the representation R whose Young tableaux has rows of length $\hat{r}_i = r_i + i - N$. In fact

$$
\sum_i r_i^2 - \frac{1}{N}\left(\sum_i r_i\right)^2 = C_2(R) + \frac{1}{12}N(N^2-1). \tag{29}
$$

The last term can be interpreted as a zero point energy, namely the energy of the lowest representation. It is proportional to the scalar curvature of the group manifold, and to the modulus squared of the vector obtained by summing the positive roots of $SU(N)$. On the other hand we have also

$$
\chi_R(\phi) = \frac{\det\{e^{ir_i\phi_j}\}}{J(\phi)}(-i)^{\frac{N(N-1)}{2}}. \tag{30}
$$

By substituting (29) and (30) into eq. (28) we obtain the modular inversion for the $SU(N)$ kernel on the cylinder, which reads

$$
\begin{aligned}
&\exp\left(-\frac{t}{24}N(N^2-1)\right)\sum_R \exp\left(-\frac{t}{2}C_2(R)\right)\chi_R(-\theta)\chi_R(\phi) \\
&= \left(\frac{N}{4\pi}\right)^{1/2} \sum_P \frac{t^{(1-N)/2}}{J(\theta)J(\phi)} \sum_{\{l_i\}} \exp\left[-\frac{1}{2t}\sum_{i=1}^N \left(\phi_i - \theta_{P(i)} + 2\pi l_i\right)^2\right],
\end{aligned} \tag{31}
$$

where the constraints $\sum_i l_i = \sum_i \phi_i = \sum_i \theta_i = 0$ are understood.

4 The Partition Function On the Torus

The kernel on a cylinder obtained in section 2 section allows one to calculate the partition function of QCD2 on a torus by simply sewing together the two ends of the cylinder, according to

$$\mathcal{Z}_{G=1}(N,t) = \int dg \mathcal{K}_2(g,g,t) = \int_0^{2\pi} \prod_i d\phi_i J^2(\phi) \mathcal{K}_2(\phi,\phi,t). \tag{32}$$

This was done in ref. [9] by using eq. (2) with $G=0$, $n=2$ and the orthogonality properties of the characters. By repeating the same calculation with our expression (23) for the kernel we are able to write $\mathcal{Z}_{G=1}(N,t)$ in terms of θ functions, whose behaviour under modular transformations is well known. The result is particularly simple in the case of the group $U(N)$ where in the expression of the kernel, eq. (23), the integers l_i and the angles θ_i and ϕ_i are unconstrained.

In this case the partition function is given by [2]

$$\begin{aligned}\mathcal{Z}_{G=1}(N,t) &= \left(\frac{t}{4\pi}\right)^{-N/2} \int_0^{2\pi} \prod_{i=1}^N d\phi_i \sum_P (-1)^P \times \\ &\times \sum_{\{l_i\}} (-1)^{(N-1)\sum_j l_j} \exp\left(-\frac{1}{2t} \sum_{i=1}^N (\phi_i - \phi_{P(i)} + 2\pi l_i)^2\right). \end{aligned} \tag{33}$$

By using the modular transformation eq. (26) for the theta function the exponentials can be made linear in the angles ϕ_i :

$$\begin{aligned}\mathcal{Z}_{G=1}(N,t) &= \int_0^{2\pi} \prod_{i=1}^N d\phi_i \sum_P (-1)^P \times \\ &\times \sum_{\{n_i\}} \exp\left(-\frac{t}{2} \sum_{i=1}^N (n_i - \delta_N)^2 + i \sum_i (n_i - \delta_N)(\phi_i - \phi_{P(i)})\right), \end{aligned} \tag{34}$$

where δ_N is 0 for odd N and 1/2 for even N. The r.h.s. of eq. (34) can be computed by the same method employed in [14] for the one-dimensional KM model, that is by decomposing each permutation into its cycles. Then

$$\mathcal{Z}_{G=1}(N,t) = \sum_{h_1,\ldots,h_N} \delta\left(\sum_{r=1}^N r h_r - N\right) (-1)^{\sum_j (j-1) h_j} \prod_{r=1}^N \left(\frac{F_r}{r}\right)^{h_r} \frac{1}{h_r!}, \tag{35}$$

where h_r is the multiplicity of a cycle of length r in a given permutation and the sum over all permutations is reproduced by summing over the h_r 's with the correct

[2]This partition function has a different zero point energy compared to the one defined for instance in ref. [9] corresponding to the overall factor $\exp[\frac{t}{2}\frac{N(N^2-1)}{12}]$

combinatorial factors. F_r is the contribution from a cycle of length r and it is given by

$$\begin{aligned} F_r &= \sum_{\{n_i\}} \int_0^{2\pi} d\phi_1 \dots d\phi_r \exp\left[-\frac{t}{2}\sum_{i=1}^{r}(n_i-\delta_N)^2 + i\sum_{i=1}^{r}(n_i-\delta_N)(\phi_i-\phi_{i+1})\right] \\ &= (2\pi)^r \sum_{n=-\infty}^{+\infty} e^{-\frac{t\,r}{2}(n-\delta_N)^2} = (2\pi)^r \theta_{\sigma(N)}\left(0, \tau = i\frac{tr}{2\pi}\right), \end{aligned} \tag{36}$$

where $\theta_{\sigma(N)}$ denotes Jacobi's θ_2 for N even and θ_3 for N odd.

This result can be expressed in a rather elegant and interesting form if one consider the grand-canonical partition function

$$\mathcal{Z}_{G=1}(q) = \sum_N \mathcal{Z}_{G=1}(N,t) q^N \tag{37}$$

The even and odd parts of $\mathcal{Z}_{G=1}(q)$ have to be computed separately. From eq. (37) we have

$$\begin{aligned} \mathcal{Z}^{\text{even}}_{G=1}(q) &= \frac{1}{2}\exp\left\{\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\theta_2\left(0, \tau = i\frac{t\,r}{2\pi}\right)(2\pi q)^r\right\} + (q \leftrightarrow -q), \\ \mathcal{Z}^{\text{odd}}_{G=1}(q) &= \frac{1}{2}\exp\left\{\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\theta_3\left(0, \tau = i\frac{t\,r}{2\pi}\right)(2\pi q)^r\right\} - (q \leftrightarrow -q). \end{aligned} \tag{38}$$

The sum over r in the exponents can be performed if one replaces the theta functions with their expressions as infinite sums, finally leading to the result

$$\begin{aligned} \mathcal{Z}^{\text{even}}_{G=1}(q) &= \frac{1}{2}\prod_{n=0}^{\infty}\left(1 + 2\pi q e^{-\frac{t}{2}\left(n+\frac{1}{2}\right)^2}\right)^2 + (q \leftrightarrow -q), \\ \mathcal{Z}^{\text{odd}}_{G=1}(q) &= \frac{1}{2}(1+2\pi q)\prod_{n=1}^{\infty}\left(1 + 2\pi q e^{-\frac{t}{2}n^2}\right)^2 - (q \leftrightarrow -q). \end{aligned} \tag{39}$$

This partition function has some similarity with the one of the one-dimensional KM model [14], in the sense that they both describe a gas of free fermions. However, in this case there are some peculiarities. The integers n can be interpreted as the discretized momenta associated with the winding of the eigenvalues, therefore the energy levels are proportional to n^2, rather than n. For $n \neq 0$ two degenerate levels are present, corresponding to winding in opposite directions. On the other hand, the integers l_i in eq. (33) are the winding numbers, namely the discretized coordinates of such modes, and the modular transformation allows us to go from the coordinate to the momentum representation. Further, the energy levels depend on the parity of the total number of fermions, which is connected with the fact that a fermionic wave function picks up a different sign according to the number of other fermions encountered in the winding. This is the meaning of the phase factor $(-1)^{(N-1)\sum_i l_i}$ in eq. (23).

We also emphasize that $\mathcal{Z}_{G=1}(q)$ has been defined keeping $t = \tilde{g}^2 \mathcal{A}$ independent of N, so eq. (39) cannot be used as such to calculate the large N limit of $\mathcal{Z}_{G=1}(N,t)$.

At least for N odd, eq. (38) has a simple connection with QED2, as one can see by remembering that $\theta_3(0, i\, t/2\pi)$ is the partition function of QED2 on a surface of area $\mathcal{A}$ [9]. Consider now QCD2 on a torus and go into the gauge where $F(x,t)$ is diagonal. An easy calculation shows that the Vandermonde determinants resulting from diagonalizing $F(x,\tau)$ are exactly cancelled by the functional integral over the off-diagonals elements of $A_\mu(x,\tau)$. This leaves one apparently with N copies of QED. However, as one goes with continuity from $x=0$ to $x=2\pi$ the eigenvalues of F will be permuted in an arbitrary way, and the sum over all permutations is needed to recover the whole partition function. One should notice finally that, in the decomposition of each permutation into irreducible cycles, each cycle of length r corresponds to a QED2 where we have to wind r times in the x direction before we go back to the original point, so in fact a QED2 on a torus of area $r\mathcal{A}$.

Consider now the case of $SU(N)$. The partition function for the group $SU(N)$ is the same as eq. (33), but with N replaced by $N-1$ and the constraint $\sum \phi_i = \sum l_i = 0$. Thus

$$\begin{aligned}\mathcal{Z}^{SU(N)}_{G=1}(N,t) &= \left(\frac{t}{4\pi}\right)^{\frac{1-N}{2}} \int_0^{2\pi} d\phi_1 \dots d\phi_N \delta\left[\sum_i \phi_i\right] \sum_P (-1)^P \int_0^{2\pi} \frac{d\beta}{2\pi} \sum_{\{l_i\}} \\ &\quad \times \exp\left\{-\frac{1}{2t}\sum_{i=1}^N (\phi_i - \phi_{P_i} + 2\pi l_i)^2 + i\beta \sum_{i=1}^N l_i\right\}. \end{aligned} \tag{40}$$

The sum over the l_i can be done by reconstructing the square and using eq. (26), in analogy to what was done in in the case of kernel of the cylinder. At this point the integral over the ϕ_i is trivial and gives

$$\begin{aligned}\mathcal{Z}^{SU(N)}_{G=1}(N,t) &= \left(\frac{t}{4\pi}\right)^{1/2} \sum_{\{n_i\}} \sum_P (-1)^P \prod_{i=1}^N \delta_{n_i n_{P_i}} \exp\left\{-\frac{t}{2}\left[\sum_{i=1}^N n_i^2 - \frac{(\sum n_i)^2}{N}\right]\right\} \\ &\times \int_0^{2\pi} d\beta \exp\left\{-\frac{Nt}{8\pi^2}\left(\beta - \frac{2\pi}{N}\sum n_i\right)\right\}. \end{aligned} \tag{41}$$

It is not difficult now to show that this partition function coincides (apart from the zero point energy) with the one of ref. [7, 8, 9],

$$\mathcal{Z}_{G=1}(N,t) = \sum_R e^{-\frac{t}{2}C_2(R)}. \tag{42}$$

In fact the determinant of δ functions in eq. (41) just forbids any couple of n_i's to take the same value and the integral over β becomes trivial after the shift of the n_i's leading to eqs. (27) and (28).

Starting from eq. (41) it is possible, following the steps leading to eqs. (37) and (39), to write in compact form the grand canonical partition function for $SU(N)$, as

$$\begin{aligned}\mathcal{Z}^{SU(N)}_{G=1}(q) &= \sum_N \mathcal{Z}^{SU(N)}_{G=1}(N,t)q^N \\ &= \left(\frac{t}{4\pi}\right)^{1/2}\int_0^{2\pi} d\beta \prod_{n=-\infty}^{+\infty}\left(1+qe^{-\frac{t}{2}\left(n-\frac{\beta}{2\pi}\right)^2}\right). \end{aligned} \tag{43}$$

The interpretation of this formula is the following: the constraint $\sum_i l_i = 0$ means that for $SU(N)$ the wave function of the center of mass of the eigenvalues is completely localized, and therefore the corresponding momentum is undetermined. This is the reason of the integration over β in eq. (43), and of the invariance of the Casimir in eq. (29) under common shifts of the integers labelling the representations.

5 Conclusions

We have shown in this paper, by means of dimensional reduction techniques, that QCD2 on a cylinder or a torus is exactly a one dimensional matrix model of the type proposed by Kazakov and Migdal, with the substantial new feature that the eigenvalues of the matter fields live on a circle. This is in agreement with the interpretation of the scalar fields as the logarithm of the Polyakov loop in the compactified dimension. We prove also that the fundamental constituents of the theory are free fermion excitations corresponding to the winding of the eigenvalues around their target space, with energy growing quadratically with the winding number. As shown by the standard form of the partition functions, eq. (42), the states of the theory are labelled by the irreducible representations of $U(N)$ (or $SU(N)$), and are states of N such free fermions.

The interpretation of QCD2 as a matrix model is natural in a formulation where the kernel on the cylinder and the partition function on the torus are expressed as an expansion in exponentials of $\frac{1}{\tilde{g}^2\mathcal{A}}$ rather than in exponentials of $\tilde{g}^2\mathcal{A}$ as in the representation expansion. From the mathematical point of view the bridge, between the two representations is a modular transformation that generalizes to the cylinder and to the torus the result already known for the disk [10] .

It would be very interesting to generalize this result to kernels with a higher number of entries and to arbitrary genus, that is to find for

$$\mathcal{K}_{G,n}(g_1,g_2,\ldots,g_n;N,t) = \sum_R d_R^{2-2G-n}\chi_R(g_1)\ldots\chi_R(g_n)e^{-\frac{t}{2}C_2(R)} \tag{44}$$

a representation in terms of exponentials of $\frac{1}{\tilde{g}^2\mathcal{A}}$. This would presumably amount to describe such kernels in terms of one dimensional matrix models whose target space includes both branching points and loops. There is one obvious difficulty in such program: by sewing two kernels together one should obtain a new kernel that depends only on the total area. This is quite natural in (44), as the exponentials are additive in the area, but in principle much more difficult with expressions containing exponentials of $\frac{1}{\tilde{g}^2\mathcal{A}}$.

6 Appendix A

In this appendix we will provide an alternative derivation of the $SU(N)$ partition function on the cylinder, which follows closely the method employed by Itzykson and Zuber [12] to derive their well-known formula for the angular integral in hermitian matrix models.

First we note, following [9], that the partition function on the cylinder can easily be written in terms of the one on the disk, thanks to the invariance under area-preserving diffeomorphisms. One just has to deform the circle into a rectangle, decompose the Wilson loop in the product of four Wilson lines, and then identify two opposite sides. Consider then the $SU(N)$ kernel on the disk, as given by [10], before periodicity is imposed:

$$\mathcal{K}_1(\phi_i;t) = \mathcal{N}(t)\prod_{i<j}\frac{\phi_i-\phi_j}{2\sin\frac{\phi_i-\phi_j}{2}}\exp\left[-\frac{1}{2t}\sum_i\phi_i^2\right]. \tag{45}$$

We want to calculate the integral

$$\mathcal{K}_2(\phi_1^i,\phi_2^i;t) = \int_{SU(N)} dh\ \mathcal{K}_1(\Lambda_1 h\Lambda_2 h^{-1};t), \tag{46}$$

where $\Lambda_1 = diag(e^{i\phi_1^i}), \Lambda_2 = diag(e^{i\phi_2^i})$, and $\mathcal{K}_1$ is a function of the eigenvalues $e^{i\phi_U^i}$ of the unitary matrix $U = \Lambda_1 h\Lambda_2 h^{-1}$. To this end, consider an arbitrary solution of the heat equation on the group, subject to the condition that it should be a class function, and a symmetric function of the eigenvalues $e^{i\phi_U^i}$. Denote this function by $f(\phi_U^i;t)$. Then

$$\begin{aligned} f(\phi_U^i;t) &= \int_{SU(N)} dg\ \mathcal{K}_1(Ug^{-1},t)f(g,0) \\ &= \int_{SU(N)} dg\ \mathcal{K}(U,g;t)f(g;0). \end{aligned} \tag{47}$$

Following standard techniques, we integrate first over the eigenvalues of g, obtaining

$$f(\phi_U^i;t) = C_N\int_{SU(N)} dS\int d\Lambda_g J^2(\Lambda_g)\mathcal{K}(\Lambda_U, S\Lambda_g S^{-1};t)f(\Lambda_g;0), \tag{48}$$

where

$$J(\Lambda_g) = i\prod_{i<j}2\sin\frac{\phi_{(g)}^i-\phi_{(g)}^j}{2}. \tag{49}$$

Consider now the antisymmetric function of Λ_U

$$\xi(\phi_U^i,t) \equiv \Delta(\Lambda_U)f(\phi_U^i,t). \tag{50}$$

We find

$$\xi(\phi^i_U;t) = \int d\Lambda_g \xi(\phi^i_g, 0) \hat{\mathcal{K}}(\Lambda_g, \Lambda_U; t), \tag{51}$$

where the kernel for ξ is

$$\hat{\mathcal{K}}(\Lambda_g, \Lambda_U; t) = C\Delta(\Lambda_g)\Delta(\Lambda_U) \int_{SU(N)} dS \mathcal{K}(\Lambda_U, S\Lambda_g S^{-1}; t). \tag{52}$$

Since $f(\phi^i_U, t)$ is a solution of the heat equation , $\xi(\phi^i_U, t)$ obeys

$$\frac{\partial \xi}{\partial t} = \Delta(\phi_U)\hat{\Delta}_U f(\phi_U; t), \tag{53}$$

where $\hat{\Delta}_U$ is the "radial" part of the Laplace-Beltrami operator on the group manifold, given by [10]

$$\hat{\Delta}_U = \frac{1}{2J} \sum_i \frac{\partial^2}{\partial \phi_i^2} \ J + \frac{1}{2} \ R_N, \tag{54}$$

with $R_N = \frac{1}{12} N(N^2 - 1)$. The function $\xi(\phi^i_U, t)$ thus obeys the simple diffusion equation

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \sum_i \frac{\partial^2}{\partial \phi_i^2} \ \xi + \frac{1}{2} R_N \ \xi. \tag{55}$$

For $R_N = 0$, ξ is a solution of a diffusion equation which is totally antisymmetric in its arguments ϕ_i, so the corresponding kernel must be the antisymmetric gaussian

$$\begin{aligned} \hat{\mathcal{K}}(\phi_1, \phi_2; t) &= \frac{1}{(2\pi t)^{\frac{N-1}{2}}} \frac{1}{N!} \sum_P (-1)^P \exp\left[-\frac{1}{2t} \sum_i \left(\phi_i^{(1)} - \phi_{P(i)}^{(2)} \right)^2 \right] \\ &= \frac{1}{(2\pi t)^{\frac{N-1}{2}}} \det\left[\exp\left(-\frac{1}{2t} (\phi_{1,i} - \phi_{2,j})^2 \right) \right]. \end{aligned} \tag{56}$$

Equating eq. (56) and eq. (52) we find that our integral is given by

$$\begin{aligned} \mathcal{K}_2(\phi^i_1, \phi^i_2; t) &= \int_{SU(N)} dh \mathcal{K}(\Lambda_1, h\Lambda_2 h^{-1}; t) \\ &= \mathcal{N}_2(t) \frac{1}{J(\Lambda^i_1) J(\Lambda^i_2)} \det\left[\exp\left(-\frac{1}{2t} (\phi_{1,i} - \phi_{2,j})^2 \right), \right] \end{aligned} \tag{57}$$

and the only modification due to the R_N term is an extra factor of $e^{R_N t/2}$, which is also present in [10].

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Note

While writing this paper we received two independent papers [15, 16] with new results on QCD2, that partially overlap our work. In particular the work of Minahan and Polychronakos [15] was of some help to us in the interpretation of the grand canonical partition functions.

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